THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH 2050A extra Tutorial

1. Show that
$$\lim \sqrt[n]{n!} = +\infty$$

2. Prove that if $\lim z_n = A$, then:
$$\lim_{n\to\infty} \frac{z_1 + z_2 + \cdots + z_n}{n} = A$$

3. Let (a_n) be a bounded sequence of real numbers. For each $n \in \mathbb{N}$, define
$$b_n = \sup_{k\geq n} a_k = \sup\{a_k : k \geq n\}, \qquad c_n = \inf_{k\geq n} a_k = \inf\{a_k : k \geq n\}.$$

   (a) Show that
   $$\lim b_n = \inf b_n, \quad \textit{and} \quad \lim c_n = \sup c_n.$$

   (b) Define the limit superior $\overline{\lim}\, a_n$ and the limit inferior $\underline{\lim}\, a_n$ of (a_n) by
   $$\overline{\lim}\, a_n = \lim b_n = \inf_{n\geq 1} \sup_{k\geq n} a_n, \quad \underline{\lim}\, a_n = \lim c_n = \sup_{n\geq 1} \inf_{k\geq n} a_n.$$

   Show that $\underline{\lim}\, a_n \leq \overline{\lim}\, a_n$.
   Remark: This is another definitions of limit inferior and limit superior, which is equivalent to the definitions on the book.

4. Let (a_n) and (b_n) be bounded sequences such that $a_n \leq b_n$ for all $n \in \mathbb{N}$. Show that
$$\overline{\lim}\, a_n \leq \overline{\lim}\, b_n, \quad \textit{and} \quad \underline{\lim}\, a_n \leq \underline{\lim}\, b_n.$$

5. Let (a_n) be a bounded sequence of real numbers. Let $\alpha = \overline{\lim}\, a_n$ and $\beta = \underline{\lim}\, a_n$.

   (a) i. Show that for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $a_n < \alpha + \epsilon$.
       ii. Show that for all $\epsilon > 0$, for all $n \in \mathbb{N}$, there exists $k \geq n$ such that $\alpha - \epsilon < a_k$.

   (b) i. Show that for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $\beta - \epsilon < a_n$.
       ii. Show that for all $\epsilon > 0$, for all $n \in \mathbb{N}$, there exists $k \geq n$ such that $a_k < \beta + \epsilon$.

6. Let (a_n) be a bounded sequence of real numbers. Define

$$E := \{x \in \mathbb{R} : \text{there is a subsequence } (a_{n_k}) \text{ of } (a_n) \text{ such that } a_{n_k} \to x\}.$$

   Let $\alpha = \overline{\lim}\, a_n$. Show that $\alpha \in E$ and $\alpha = \sup E$.
   Remark: This result also holds for limit inferior.

7. Let (a_n) be a bounded sequence of real numbers. Show that (a_n) is convergent if and only if $\overline{\lim}\, a_n = \underline{\lim}\, a_n$. In this case, we have $\lim a_n = \overline{\lim}\, a_n = \underline{\lim}\, a_n$.

8. Prove that convergence of (s_n) implies convergence of $(|s_n|)$. Is the converse true?

9. if $s_1 = \sqrt{2}$, and

$$s_{n+1} = \sqrt{2 + s_n} \qquad (n = 1, 2, 3, ...),$$

   prove that (s_n) converges, and that $s_n < 2$ for $n = 1, 2, 3, ...$

10. For any two sequence (a_n), (b_n) of real number, prove that

$$\overline{\lim}(a_n + b_n) \leq \overline{\lim}\, a_n + \overline{\lim}\, b_n,$$

    provided that $\overline{\lim}\, a_n, \overline{\lim}\, b_n \neq +\infty$.

11. Fix a positive number α, choose $x_1 > \sqrt{\alpha}$, and define x_{n+1} by following formula

$$x_{n+1} = \frac{1}{2}(x_n + \frac{\alpha}{x_n}).$$

    Prove that (x_n) convergence and that $\lim x_n = \sqrt{\alpha}$.

12. If you have any question about midterm, you can email me via iauyeung@math.cuhk.edu.hk or you can come to 222C to ask me on Monday before 4:30pm.